POLYNOMIALS THAT HAVE GOLDEN RATIO ZEROS

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Abstract. When the golden ratio and its conjugate are zeros to a polynomial, two of the coefficients are functions of the Fibonacci sequence in terms of the other coefficients, which characterize the polynomials completely. These functions are used to derive some F_n, L_n, and golden ratio identities. In many cases, this is generalized to the Lucas functions U_n and V_n, with associated quadratic root pair. A new type of geometric progression is introduced.

1. Introduction

Draim and Bicknell in [4] observed that the polynomials $x^2 - L_n x + (-1)^n$ have the zeros α^n and β^n, where α and β are the golden ratio and its conjugate, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, and L_n is the Lucas number sequence, [10]. This same fact was presented in more detail by Hoggatt in [8, pp. 66–67]. In Alexanderson's solution [1] to Wall's proposed problem [19], it was shown that the polynomials $x^n - F_n x - F_{n-1}$ have α (and β) as zeros because $x^2 - x - 1$ is a factor of these polynomials, where F_n is the Fibonacci number sequence. He and Shiue noticed in [7] that α and β are zeros of the polynomials $x^n - F_n x - F_{n-1}$ by observing the identities $\alpha^n = \alpha F_n + F_{n-1}$ and $\beta^n = \beta F_n + F_{n-1}$, which were obtained in [7] as applications of their degree reduction approach and were also proved using mathematical induction by Thoro in [17].

Inspired by all of the above results, in this paper a general rule that uses Fibonacci and Lucas number sequences is presented to recognize and determine the coefficients of polynomials that have both $r\alpha^n$ and $r\beta^n$ ($r \neq 0$) as zeros. Furthermore, the polynomials with zeros $r\alpha^n$ and $r\beta^n$ are completely characterized by their constant term and linear term coefficients. This relationship is used to derive many new and well-known identities involving F_n, L_n, and the golden ratio. In the process of writing this paper, we realized that, with little modification, many of the formulas may be made to apply, not only to the golden ratio and its conjugate, but to any pair of quadratic roots, a and b, and associated number sequences, described by Lucas in [10], such as the Pell sequence ($a, b = 1 \pm \sqrt{2}$), and a Fermat sequence ($a = 2$, $b = 1$). Lucas denoted a and b as the two roots of the equation $x^2 = px - q$, and described some useful number sequences generated by the simple periodic numerical functions $U_n = \frac{a^n - b^n}{a-b}$, and $V_n = a^n + b^n$, [10, eqns. (2), p. 2]. The special case of $p = 1$, and $q = -1$ gives us $a = \alpha$, $b = \beta$, $U_n = F_n$, and $V_n = L_n$. Wherever the general case is applicable, we use the symbols a and b for the zeros, in place of α and β, and U_n and V_n for the number sequence element symbols, in place of F_n and L_n. The paper has the following structure: in Section 2 we present the coefficient characterization of polynomials having ra^n and rb^n ($r \neq 0$) as zeros. Equivalent expressions of this characterization, and examples, are then presented in Section 3. The application of the coefficient characterization in the construction of identities involving F_n, L_n, and the golden ratio, generalized where feasible to U_n, V_n, and a, b, will be given in Section 4.

2. Coefficient characterization of polynomials having ra^n and rb^n zeros

Proposition 2.1. *Denote*

$$P_n(x) = \sum_{i=0}^{n} c_i x^i = c_0 + c_1 x + c_2 x^2 + \ldots + c_{n-1}x^{n-1} + c_n x^n \,, \quad n \geq 2, \tag{2.1}$$

a and b are the two roots of the equation $q - px + x^2 = 0$, specifically

$$a = \frac{p + \sqrt{p^2 - 4q}}{2}, \quad \textit{and} \quad b = \frac{p - \sqrt{p^2 - 4q}}{2},$$

$$U_n = \begin{cases} \frac{a^n - b^n}{a-b} & if\ a \neq b \\ n\left(\frac{p}{2}\right)^{n-1} & if\ a = b\,, \end{cases} \qquad \text{(Lucas Binet Formula)}$$

$$\textit{and} \quad V_n = a^n + b^n, \qquad \text{(Lucas Formula)}$$

then, for $r \neq 0$, $P_n(ra^k) = P_n(rb^k) = 0$ if and only if these hold:

$$c_1 = \begin{cases} -\frac{1}{U_k}\sum_{i=2}^{n} c_i r^{i-1} U_{ki} & if\ k \neq 0 \wedge a \neq b \\ -\sum_{i=2}^{n} i c_i r^{i-1} & if\ k = 0 \vee a = b \end{cases} \quad \textit{and} \tag{2.2}$$

$$c_0 = -\frac{1}{2}\sum_{i=1}^{n} c_i r^i V_{ki} = \begin{cases} \frac{1}{2}\sum_{i=2}^{n} c_i r^i \left(\frac{V_k U_{ki}}{U_k} - V_{ki}\right) & if\ k \neq 0 \wedge a \neq b \\ \sum_{i=2}^{n} (i-1) c_i r^i & if\ k = 0 \vee a = b \end{cases} . \tag{2.3}$$

Proof. If ra^k $(r \neq 0)$ is a zero of $P_n(x)$, then $P_n(x) = (x - ra^k)Q_{n-1}(x) = 0$, where

$$Q_{n-1}(x) = \sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} r^j a^{kj} x^{i-1-j} = 0 \tag{2.4}$$

is determined using polynomial division of $P_n(x)$ by $x - ra^k$. In fact, by multiplying $x - ra^k$ and $Q_{n-1}(x)$, which is shown in (2.4), we may reconstruct $P_n(x)$ as follows.

$$\begin{aligned} (x - ra^k)Q_{n-1}(x) &= (x - ra^k)\sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} r^j a^{kj} x^{i-1-j} \\ &= \sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} r^j a^{kj} x^{i-j} - \sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} r^{j+1} a^{k(j+1)} x^{i-1-j} \\ &= \sum_{i=1}^{n} c_i x^i + \sum_{i=2}^{n} c_i \sum_{j=1}^{i-1} r^j a^{kj} x^{i-j} - \sum_{i=1}^{n} c_i r^i a^{ki} - \sum_{i=2}^{n} c_i \sum_{j=1}^{i-1} r^j a^{kj} x^{i-j} \\ &= \sum_{i=1}^{n} c_i x^i + c_0 = P_n(x), \end{aligned}$$

where we use $c_0 = -\sum_{i=1}^{n} c_i r^i a^{ki}$ in the last line, because $P_n(ra^k) = 0$. In the third line, The $j = 0$ terms are separated from the left sum of line two, and the $j = i - 1$ terms are separated from the right sum of line two.

If both ra^k and rb^k are roots of $P_n(x) = 0$, then $Q_{n-1}(rb^k) = 0$. Hence, substituting $x = rb^k$ into (2.4) and solving for c_1 yields

$$c_1 = -\sum_{i=2}^{n} c_i \sum_{j=0}^{i-1} r^j a^{kj} r^{i-1-j} b^{k(i-1-j)} = -\sum_{i=2}^{n} c_i r^{i-1} \frac{a^{ki} - b^{ki}}{a^k - b^k}, \; k \neq 0 \tag{2.5}$$

where the last equality is obtained by using the identity, [2, p. 6],

$$\sum_{j=0}^{i-1} x^j y^{i-1-j} = \frac{x^i - y^i}{x - y} .$$

Utilizing the Lucas Binet formula $U_n = (a^n - b^n)/a - b$, one may write the ratio on the rightmost side of (2.5) as U_{ki}/U_k, which implies that (2.5) is equivalent to (2.2) for the case of $k \neq 0$. If $k = 0$, then the first equation of (2.5) gives $c_1 = -\sum_{i=2}^{n} i c_i r^{i-1}$, which implies (2.2) in the case of $k = 0$.

Next, substituting ra^k and rb^k for x into $P_n(x) = 0$, and then solving for c_0, we obtain, respectively,

$$c_0 = -\sum_{i=1}^{n} c_i r^i a^{ki} \quad and \quad c_0 = -\sum_{i=1}^{n} c_i r^i b^{ki}.$$

Hence,

$$c_0 = -\sum_{i=1}^{n} c_i r^i \, \frac{a^{ki} + b^{ki}}{2}.$$

Utilizing $a^n + b^n = V_n$, we may re-write this as the first equation of (2.3). For $k \neq 0$, substituting the corresponding expression of c_1 shown in (2.2) into the first equation of (2.3) yields

$$\begin{aligned} c_0 &= -\frac{1}{2}\left(c_1 r V_k + \sum_{i=2}^{n} c_i r^i V_{ki} \right) \\ &= \frac{1}{2} \sum_{i=2}^{n} c_i r^i \left(\frac{V_k U_{ki}}{U_k} - V_{ki} \right). \end{aligned} \tag{2.6}$$

Similarly, for $k = 0$, this holds:

$$c_0 = -\frac{1}{2}\sum_{i=1}^{n} c_i r^i V_0 = -rc_1 - \sum_{i=2}^{n} c_i r^i = \sum_{i=2}^{n} (i-1) c_i r^i.$$

For the case of $a = b$, if $a = b$, then $ra^k = rb^k$, i.e., there is a duplicate zero. Such is also the case when $k = 0$. Suppose $P_n(ta^m) = P_n(tb^m) = 0$, and that $a = b$. Let $r = ta^m$, and $k = 0$, then $ra^k = ta^m$, and it seen that the case of $a = b$ is the same as the case of $k = 0$.

This completes the proof that if $P_n(ra^k) = P_n(rb^k) = 0$, then c_1 has the value given by equation (2.2), and c_0 has the value given by equation (2.3). Conversely, if the polynomial $P_n(x)$ denoted by (2.1) possesses the coefficients c_1 and c_0 presented as (2.2) and (2.3), respectively, then we may substitute the expression of c_1 into (2.3) to obtain (2.6). Using equations (2.6) and (2.2) for $k \neq 0$, we can substitute them for c_0 and c_1, respectively, into $P_n(x) = c_0 + c_1 x + \sum_{i=2}^n c_i x^i$ to get:

$$P_n(x) = \sum_{i=2}^n c_i r^i \left(\frac{V_k}{2} \frac{U_{ki}}{U_k} - \frac{V_{ki}}{2} \right) - \left(\sum_{i=2}^n c_i r^{i-1} \frac{U_{ki}}{U_k} \right) x + \sum_{i=2}^n c_i x^i. \tag{2.7}$$

Summing and subtracting $a^n + b^n = V_n$ with $a^n - b^n = \delta U_n$, where $\delta = a - b$, we get [10, p. 8, eqns. (6)]

$$a^n = \frac{V_n + \delta U_n}{2} \qquad \textit{and} \qquad b^n = \frac{V_n - \delta U_n}{2},$$

which, for $a = \alpha$ and $b = \beta$, are the well-known formulas [3, 15, 16, 18]

$$\alpha^n = \frac{L_n + \sqrt{5} F_n}{2} \qquad \textit{and} \qquad \beta^n = \frac{L_n - \sqrt{5} F_n}{2}$$

Substituting the above expression of ra^k for x, and $r^i a^{ki}$ for x^i into equation (2.7) yields

$$\begin{aligned} P_n(ra^k) &= \sum_{i=2}^n c_i r^i \left(\frac{V_k}{2} \frac{U_{ki}}{U_k} - \frac{V_{ki}}{2} \right) - \sum_{i=2}^n c_i r^{i-1} \frac{U_{ki}}{U_k} r \frac{V_k + \delta U_k}{2} \\ &\quad + \sum_{i=2}^n c_i r^i \frac{V_{ki} + \delta U_{ki}}{2} \\ &= \sum_{i=2}^n c_i r^i \left(\frac{V_k}{2} \frac{U_{ki}}{U_k} - \frac{V_{ki}}{2} - \frac{V_k}{2} \frac{U_{ki}}{U_k} - \frac{\delta U_{ki}}{2} + \frac{V_{ki} + \delta U_{ki}}{2} \right) = 0. \end{aligned}$$

For the case of $k = 0$, $x = r$, and it is easy to see the corresponding c_1 and c_0 shown in (2.2) and (2.3), put into $P_n(x) = c_0 + c_1 x + \sum_{i=2}^n c_i x^i$, yield

$$P_n(r) = \sum_{i=2}^n (i-1) c_i r^i - \sum_{i=2}^n i c_i r^{i-1} r + \sum_{i=2}^n c_i r^i = 0.$$

Similarly, we may prove $P_n(rb^k) = 0$, which shows that both ra^k and rb^k are the zeros of $P_n(x)$ if its coefficients c_1 and c_0 are given as (2.2) and (2.3), respectively. This completes the proof of the proposition.

□

Remark Here is a proof that $U_n = n \left(\frac{p}{2}\right)^{n-1}$ when $a = b$.

Proof. Using the identity $a^n - b^n = (a - b) \sum_{i=0}^{n-1} a^i b^{n-1-i}$ we have

$$U_n = \frac{a^n - b^n}{a - b} = \frac{a - b}{a - b} \sum_{i=0}^{n-1} a^i b^{n-1-i} = \sum_{i=0}^{n-1} a^i b^{n-1-i}\ . \tag{2.8}$$

When $a = b$, $p^2 = 4q$, and $\frac{p}{2} = a = b$. Substituting $\frac{p}{2}$ for both a and b in this last expression of U_n yeilds

$$U_n = \sum_{i=0}^{n-1} \left(\frac{p}{2}\right)^i \left(\frac{p}{2}\right)^{n-1-i} = \sum_{i=1}^{n} \left(\frac{p}{2}\right)_i^{n-1} = n \left(\frac{p}{2}\right)^{n-1}.$$

□

Let $r = h^k$ ($h \neq 0$). From Proposition 2.1:

Corollary 2.2. *If $P_n(x)$, a, b, U_n, and V_n are defined as in Proposition 2.1, then for $h \neq 0$, $P_n((ha)^k) = P_n((hb)^k) = 0$ if and only if these hold:*

$$c_1 = \begin{cases} -\frac{1}{U_k}\sum_{i=2}^n c_i h^{k(i-1)} U_{ki} & if\ k \neq 0 \wedge a \neq b \\ -\sum_{i=2}^n i c_i & if\ k = 0 \vee a = b \end{cases} \quad and \tag{2.9}$$

$$c_0 = -\frac{1}{2}\sum_{i=1}^n c_i h^{ki} V_{ki} = \begin{cases} \frac{1}{2}\sum_{i=2}^n c_i h^{ki}\left(\frac{V_k U_{ki}}{U_k} - V_{ki}\right) & if\ k \neq 0 \wedge a \neq b \\ \sum_{i=2}^n (i-1) c_i & if\ k = 0 \vee a = b \end{cases}. \tag{2.10}$$

If $h = 1$, then a particular case of Corollary 2.2 can be written as

Corollary 2.3. *Let $P_n(x)$, a, b, U_n, and V_n be denoted as in Proposition 2.1, then $P_n(a^k) = P_n(b^k) = 0$ if and only if these hold:*

$$c_1 = \begin{cases} -\frac{1}{U_k}\sum_{i=2}^n c_i U_{ki} & if\ k \neq 0 \wedge a \neq b \\ -\sum_{i=2}^n i c_i & if\ k = 0 \vee a = b \end{cases} \quad and \tag{2.11}$$

$$c_0 = -\frac{1}{2}\sum_{i=1}^n c_i V_{ki} = \begin{cases} \frac{1}{2}\sum_{i=2}^n c_i \left(\frac{V_k U_{ki}}{U_k} - V_{ki}\right) & if\ k \neq 0 \wedge a \neq b \\ \sum_{i=2}^n (i-1) c_i & if\ k = 0 \vee a = b \end{cases}. \tag{2.12}$$

Similarly, $h = -1$ gives us the following particular case of Corollary 2.2.

Corollary 2.4. *Let $P_n(x)$, a, b, U_n, and V_n be denoted as in Proposition 2.1, then $P_n((-a)^k) = P_n((-b)^k) = 0$ if and only if these hold:*

$$c_1 = \begin{cases} -\frac{1}{U_k}\sum_{i=2}^n c_i (-1)^{k(i-1)} U_{ki} & if\ k \neq 0 \wedge a \neq b \\ -\sum_{i=2}^n i c_i & if\ k = 0 \vee a = b \end{cases} \quad and \tag{2.13}$$

$$c_0 = -\frac{1}{2}\sum_{i=1}^n c_i (-1)^{ki} V_{ki} = \begin{cases} \frac{1}{2}\sum_{i=2}^n c_i (-1)^{ki}\left(\frac{V_k U_{ki}}{U_k} - V_{ki}\right) & if\ k \neq 0 \wedge a \neq b \\ \sum_{i=2}^n (i-1) c_i & if\ k = 0 \vee a = b\,. \end{cases} \tag{2.14}$$

A property of quadratic roots is that $a^{-1} = q^{-1}b$. There is an interesting property of polynomial equations that will be combined with this property, and Proposition 2.1, to derive more identities in the next section. We present this property in a lemma, because we have not been able to find it published, though it is related to reciprocal polynomials, of which an application is discussed in [5, p. 250].

Lemma 2.5. *Let $P_n(x) = \sum_{i=0}^n c_i x^i$, $R_n(y) = \sum_{i=0}^n d_i y^i$, and $d_i = c_{n-i}$ for all i. If $z_1, z_2, \ldots, z_n$ are zeros of $P_n(x)$, then the zeros of $R_n(y)$ are $\frac{1}{z_1}, \frac{1}{z_2}, \ldots, \frac{1}{z_n}$. In other words, if the order of the coefficients of the terms of a polynomial equation in standard form are reversed, then the roots of the new polynomial are the reciprocals of the roots of the original polynomial.*

Proof. Multiply $R_n(y) = 0$ by $\frac{1}{y^n}$ to get

$$c_n \frac{1}{y^n} + c_{n-1}\frac{1}{y^{n-1}} + c_{n-2}\frac{1}{y^{n-2}} + \ldots + c_1 \frac{1}{y} + c_0 = 0 = P_n\left(\frac{1}{y}\right)$$

This new polynomial equation has the same roots as $R_n(y) = 0$, because multiplying both sides of the equation by the same value will not change the roots to the equation. Now, let $x = \frac{1}{y}$, producing $P_n(x) = 0$, which was given to have roots $z_1, z_2, \ldots, z_n$. Since $y = \frac{1}{x}$, the values of y that make $P_n\left(\frac{1}{y}\right) = 0$ are $\frac{1}{z_1}, \frac{1}{z_2}, \ldots, \frac{1}{z_n}$. $R_n(y) = 0$ will have these same roots.

□

We may use this lemma to characterize polynomials $P_n(x)$ having ra^k and rb^k as zeros by their c_n and c_{n-1} coefficients:

Corollary 2.6. *Denote $P_n(x)$, a, b, p, q, U_n, and V_n as in Proposition 2.1, then for $r \neq 0$, $P_n(ra^k) = P_n(rb^k) = 0$ if and only if these hold:*

$$c_{n-1} = \begin{cases} -\frac{1}{U_k}\sum_{i=2}^{n} c_{n-i}(q^k r)^{1-i} U_{ki} & if\ k \neq 0 \wedge a \neq b \\ -\sum_{i=2}^{n} i c_{n-i} r^{1-i} & if\ k = 0 \vee a = b \end{cases} \quad and \tag{2.15}$$

$$c_n = -\frac{1}{2}\sum_{i=1}^{n} c_{n-i}(q^k r)^{-i} V_{ki} = \begin{cases} \frac{1}{2}\sum_{i=2}^{n} c_{n-i}(q^k r)^{-i}\left(\frac{V_k U_{ki}}{U_k} - V_{ki}\right), & k \neq 0 \wedge a \neq b \\ \sum_{i=2}^{n}(i-1) c_{n-i} r^{-i}, & k = 0 \vee a = b\,. \end{cases} \tag{2.16}$$

Proof. Let $R_n(y) = \sum_{i=0}^{n} d_i x^i$ have zeros ta^k, tb^k. The value of d_1 may be determined using equation (2.2), and d_0 using equation (2.3), where r in the equations is t. Now, suppose that $d_i = c_{n-i}$ for all i, and $t = q^{-k}r^{-1}$, then by lemma 2.5, and $a^{-1} = q^{-1}b$, the roots to $P_n(x) = \sum_{i=0}^{n} c_i x^i = 0$ are ra^k and rb^k. $c_{n-1} = d_1$, and $c_n = d_0$. Replacing c_1 with c_{n-1}, c_0 with c_n, and r with $q^{-k}r^{-1}$ in equations (2.2) and (2.3), converts them into equations (2.15) and (2.16). □

3. Equivalent Expressions

Conditions presented in the proposition above can be used to generate additional expressions. However, in order to present them in generalized form, we must first present a few simple identities involving U_n and V_n.

Denote $P_n(x)$, a, b, p, q, U_n, and V_n as in Proposition 2.1, then

$$V_n = U_{n+1} - qU_{n-1}\,, \tag{3.1}$$

$$V_n = pU_n - 2qU_{n-1}\,, \tag{3.2}$$

$$(A) \quad U_{-n} = -q^{-n}U_n\,, \qquad (B) \quad V_{-n} = q^{-n}V_n\,, \tag{3.3}$$

$$U_{m+n} = U_{m+1}U_n - qU_mU_{n-1}\,, \text{ and} \tag{3.4}$$

$$q^n U_{m-n} = U_m U_{n+1} - U_{m+1}U_n. \tag{3.5}$$

Proof. To prove identity (3.1), substitute into it ab for q, and the Lucas and Lucas Binet formulae equivalents for V_n and U_n:

$$a^n + b^n = \frac{a^{n+1} - b^{n+1}}{a-b} - ab\frac{a^{n-1} - b^{n-1}}{a-b}$$

Multiplying both sides of the equation by $(a-b)$, and distributing we have

$$a^{n+1} + ab^n - a^n b - b^{n+1} = a^{n+1} - b^{n+1} - a^n b + ab^n$$

The LHS and RHS of this equation are now obviously equal, then likewise must be those of the original equation.

Identity (3.2) is obtained by substituting into identity (3.1), $pU_n - qU_{n-1}$ for U_{n+1}, which comes from [10, eqns. (10), p. 10].

Equations (3.3)(A) and (B) are easily found utilizing $a^{-n} = q^{-n}b^n$ in the Lucas Binet and Lucas formulas.

To prove Identity (3.4), Note that $U_0 = 0$, $U_1 = 1$, and $U_{-1} = -1/q$. Using these, it is easily verified that the identity is true for $m = 0$, and $m = 1$ (for any value of n). Let $\mathbf{n} = n + m$, then $U_{n+m+1} = U_{\mathbf{n}+1}$, for which, we just verified, the identity is true (the case of $m = 1$). Therefore, it is true for $m = 2$, and by induction for $m = 3, 4, \ldots$ and so on.

Identity (3.5) is a consequence of Identity (3.4) with Identity (3.3)(A). It is interesting that, given two U_i to use in this identity, it does not matter which of the two is assigned as U_n.

□

Note that when $p = 1$, and $q = -1$, equations (3.1) to (3.3) become the well known Fibonacci and Lucas number identities, $L_n = F_{n+1} + F_{n-1}$, $L_n = F_n + 2F_{n-1}$, $F_{-n} = (-1)^{n+1}F_n$, and $L_{-n} = (-1)^n L_n$ (see, for example, [8, pp. 27, 28]). Compare identities (3.4) and (3.5) to identities (8) and (9) in [18, p. 176].

The following corollary expresses two equivalent forms of equation (2.3).

Corollary 3.1. *Let $P_n(x)$, a, b, p, q, U_n, and V_n be denoted as in Proposition 2.1, and $P_n(ra^k) = P_n(rb^k) = 0$, then the following expressions of the coefficient c_0 are equivalent.*

$$\text{(A) } c_0 = -\frac{1}{2}\sum_{i=1}^{n} c_i r^i V_{ki},\ \text{(B) } c_0 = q\sum_{i=1}^{n} c_i r^i U_{ki-1},\ \text{(C) } c_0 = \frac{q^k}{U_k}\sum_{i=2}^{n} c_i r^i U_{k(i-1)},\quad k \neq 0 \tag{3.6}$$

Equivalent forms of (2.10) *are*

$$\text{(A) } c_0 = q\sum_{i=1}^{n} c_i h^{ki} U_{ki-1} \quad \textit{and} \quad \text{(B) } c_0 = \frac{q^k}{U_k}\sum_{i=2}^{n} c_i h^{ki} U_{k(i-1)}\,, \qquad k \neq 0 \tag{3.7}$$

when both $(ha)^k$ and $(hb)^k$ are zeros of $P_n(x)$, respectively.

If $h = 1$, then equation (2.12) *is equivalent to:*

$$\text{(A) } c_0 = q\sum_{i=1}^{n} c_i U_{ki-1} \quad \textit{and} \quad \text{(B) } c_0 = \frac{q^k}{U_k}\sum_{i=2}^{n} c_i U_{k(i-1)}\,, \qquad k \neq 0 \tag{3.8}$$

Similarly, for the case of $h = -1$, equation (2.14) *is equivalent to:*

$$\text{(A) } c_0 = q\sum_{i=1}^{n} c_i (-1)^{ki} U_{ki-1} \quad \textit{and} \quad \text{(B) } c_0 = \frac{q^k}{U_k}\sum_{i=2}^{n} c_i (-1)^{ki} U_{k(i-1)}\,, \qquad k \neq 0 \tag{3.9}$$

Proof. (3.6)(A) is the leftmost equation of (2.3). The equivalence between (3.6)(A) and Equation (3.6)(B) is proved by beginning with (2.2) and manipulating it. Begin by multiplying both sides of equation (2.2) by $-rpU_k$ which yields

$$-c_1 rpU_k = \sum_{i=2}^{n} c_i r^i pU_{ki}.$$

Then, a simple algebra produces

$$c_1 r\left(-pU_k + 2qU_{k-1} - 2qU_{k-1}\right) = \sum_{i=2}^{n} c_i r^i \left(pU_{ki} + 2qU_{ki-1} - 2qU_{ki-1}\right).$$

Using identity (3.2), $-pU_k + 2qU_{k-1}$ and $pU_{ki} - 2qU_{ki-1}$ are replaced with $-V_k$ and V_{ki}, respectively, which returns

$$c_1 r\left(-V_k - 2qU_{k-1}\right) = \sum_{i=2}^{n} c_i r^i \left(V_{ki} + 2qU_{ki-1}\right).$$

Distributing yields

$$-c_1 r V_k - 2qc_1 r U_{k-1} = \sum_{i=2}^{n} c_i r^i V_{ki} + \sum_{i=2}^{n} 2qc_i r^i U_{ki-1}.$$

Hence, we have

$$-2qc_1 r U_{k-1} - \sum_{i=2}^{n} 2qc_i r^i U_{ki-1} = c_1 r V_k + \sum_{i=2}^{n} c_i r^i V_{ki},$$

which implies

$$-2q \sum_{i=1}^{n} c_i r^i U_{ki-1} = \sum_{i=1}^{n} c_i r^i V_{ki}.$$

Finally, we obtain

$$q \sum_{i=1}^{n} c_i r^i U_{ki-1} = -\frac{1}{2} \sum_{i=1}^{n} c_i r^i V_{ki}.$$

By equation (3.6)(A), the RHS of the previous equation is equal to c_0 and thus the LHS is as well. Thus, equation (3.6)(B) is equivalent to (3.6)(A).

To prove equation (3.6)(C), we begin by separating the $i = 1$ term from the summation used for c_0 in equation (3.6)(B) as such:

$$c_0 = q \sum_{i=1}^{n} c_i r^i U_{ki-1} = q\left(c_1 r U_{k-1} + \sum_{i=2}^{n} c_i r^i U_{ki-1}\right).$$

Replacing c_1 with the value given in equation (2.2), we have

$$c_0 = q\left(-\frac{U_{k-1}}{U_k} \sum_{i=2}^{n} c_i r^i U_{ki} + \sum_{i=2}^{n} c_i r^i U_{ki-1}\right),$$

which implies

$$c_0 = \frac{q}{U_k} \sum_{i=2}^{n} c_i r^i \left(U_k U_{ki-1} - U_{k-1} U_{ki}\right).$$

Here we can use identity (3.5). Let $n = k - 1$ and $m = ki - 1$ in this identity, and it becomes

$$q^{k-1} U_{ki-k} = U_k U_{ki-1} - U_{k-1} U_{ki}. \tag{3.10}$$

Substituting $q^{k-1} U_{k(i-1)}$ for $U_k U_{ki-1} - U_{k-1} U_{ki}$ into our last result for c_0, we get equation (3.6)(C). Setting $r = h^k$ into (3.6), we may prove (2.10) is equivalent to (3.7)(A) and (B). For the special cases of $h = 1$ and $h = -1$, the results are obvious.

□

When $k = 1$ and $h = 1$, equations (2.11) and (3.7)(B) (and (3.7)(A) because $U_0 = 0$) become:

$$\text{(A)} \quad c_1 = -\sum_{i=2}^{n} c_i U_i \qquad\qquad \text{(B)} \quad c_0 = q\sum_{i=2}^{n} c_i U_{i-1} \tag{3.11}$$

This is illustrated in table 1 for the special case of $r = 1$, $k = 1$, $p = 1$, and $q = -1$, which leads to $a = \alpha$, and $b = \beta$.

n	Values of the coefficients c_1 and c_0 in $P_n(x)$ when $P_n(\alpha) = P_n(\beta) = 0$
2	$c_1 = -c_2$ $c_0 = -c_2$
3	$c_1 = -(c_2 + 2c_3)$ $c_0 = -(c_2 + c_3)$
4	$c_1 = -(c_2 + 2c_3 + 3c_4)$ $c_0 = -(c_2 + c_3 + 2c_4)$
5	$c_1 = -(c_2 + 2c_3 + 3c_4 + 5c_5)$ $c_0 = -(c_2 + c_3 + 2c_4 + 3c_5)$
6	$c_1 = -(c_2 + 2c_3 + 3c_4 + 5c_5 + 8c_6)$ $c_0 = -(c_2 + c_3 + 2c_4 + 3c_5 + 5c_6)$
7	$c_1 = -(c_2 + 2c_3 + 3c_4 + 5c_5 + 8c_6 + 13c_7)$ $c_0 = -(c_2 + c_3 + 2c_4 + 3c_5 + 5c_6 + 8c_7)$
8	$c_1 = -(c_2 + 2c_3 + 3c_4 + 5c_5 + 8c_6 + 13c_7 + 21c_8)$ $c_0 = -(c_2 + c_3 + 2c_4 + 3c_5 + 5c_6 + 8c_7 + 13c_8)$
9	$c_1 = -(c_2 + 2c_3 + 3c_4 + 5c_5 + 8c_6 + 13c_7 + 21c_8 + 34c_9)$ $c_0 = -(c_2 + c_3 + 2c_4 + 3c_5 + 5c_6 + 8c_7 + 13c_8 + 21c_9)$
10	$c_1 = -(c_2 + 2c_3 + 3c_4 + 5c_5 + 8c_6 + 13c_7 + 21c_8 + 34c_9 + 55c_{10})$ $c_0 = -(c_2 + c_3 + 2c_4 + 3c_5 + 5c_6 + 8c_7 + 13c_8 + 21c_9 + 34c_{10})$

TABLE 1. Values of c_1 and c_0 based on formulas (3.11)(A) and (B)

Example 3.2. *We suspect that $x = \alpha$ and $x = \beta$ are zeros of $P_4(x) = -7-10x+3x^2+2x^3+x^4$, but we want to know if they are exact zeros. This would not normally be a simple task, because α and β are irrational numbers. Referring to Table 1, at $n = 4$, we see that if $c_1 = -(c_2 + 2c_3 + 3c_4)$, and $c_0 = -(c_2 + c_3 + 2c_4)$, then $x = \alpha$ and $x = \beta$ are exact zeros of the polynomial. Testing: $c_1 = -(3 + 2 \cdot 2 + 3 \cdot 1) = -10$, and $c_0 = -(3 + 2 + 2 \cdot 1) = -7$. Therefore, α and β are indeed exact zeros.*

For comparison, we shall employ a different method to check if α and β are exact zeros of this polynomial: Utilizing the identities $\alpha^n = \frac{L_n+F_n\sqrt{5}}{2}$ and $\beta^n = \frac{L_n-F_n\sqrt{5}}{2}$, we may make the following substitutions into this equation: $\alpha = \frac{1+\sqrt{5}}{2}$ for x, $\alpha^2 = \frac{3+\sqrt{5}}{2}$ for x^2, $\alpha^3 = \frac{4+2\sqrt{5}}{2}$ for x^3, and $\alpha^4 = \frac{7+3\sqrt{5}}{2}$ for x^4. The substitution result is

$$P_4(\alpha) = -7 + \frac{-10(1+\sqrt{5}) + 3(3+\sqrt{5}) + 2(4+2\sqrt{5}) + 7 + 3\sqrt{5}}{2} = 0$$

Substitution of the conjugates of these values, which are the corresponding powers of β, for the powers of x, confirms that $P_4(\beta)$ also equals zero. The second method confirms the conclusion of the first method, that both α and β are exact zeros. Obviously, the first method using formulas (3.11)*(A) and (B), as illustrated in Table 1, is the simpler method.*

Corollary 3.3. *The following expressions of the coefficient c_n of polynomial $P_n(x) = \sum_{i=0}^{n} c_i$, having zeros ra^k and rb^k, are equivalent.*

$$\text{(A) } c_n = -\frac{1}{2}\sum_{i=1}^{n} c_{n-i}q^{-ki}r^{-i}V_{ki}, \qquad \text{(B) } c_n = \sum_{i=1}^{n} c_{n-i}q^{-(ki-1)}r^{-i}U_{ki-1},$$

$$\text{(C) } c_n = \frac{1}{U_k}\sum_{i=2}^{n} c_{n-i}q^{-k(i-1)}r^{-i}U_{k(i-1)}, \quad k \neq 0 \tag{3.12}$$

Proof. Applying lemma 2.5, we replace c_i with c_{n-i}, and r with $q^{-k}r^{-1}$ in equations (3.6)(A), (B), and (C) to generate equations (3.12)(A), (B), and (C).

□

Example 3.4. *Consider the polynomial $P_5(x) = 2x^5 - 7x^4 + 4x^3 + 3x^2 - 186x - 44$. It can be confirmed that α^3 and β^3 are exact zeros of this polynomial because, using equations* (2.11) *and* (3.8)*(B):*

$$\begin{aligned}
c_1 &= -\frac{1}{F_3}\left(c_2F_{3\cdot 2} + c_3F_{3\cdot 3} + c_4F_{3\cdot 4} + c_5F_{3\cdot 5}\right) = -\frac{1}{2}\left(3F_6 + 4F_9 - 7F_{12} + 2F_{15}\right) \\
&= -\frac{1}{2}\left(3\cdot 8 + 4\cdot 34 - 7\cdot 144 + 2\cdot 610\right) = -186 \quad \textit{and} \\
c_0 &= \frac{(-1)^3}{F_3}\left(c_2F_{3\cdot 1} + c_3F_{3\cdot 2} + c_4F_{3\cdot 3} + c_5F_{3\cdot 4}\right) = -44.
\end{aligned}$$

Using this same polynomial above as an example of corollary 2.6, using formulas (2.15) *and* (3.12)*(C), with $r = 1$, and $q = -1$:*

$$\begin{aligned}
c_{n-1} &= -\frac{1}{F_3}\left(c_3(-1)^{-3\cdot 1}F_{3\cdot 2} + c_2(-1)^{-3\cdot 2}F_{3\cdot 3} + c_1(-1)^{-3\cdot 3}F_{3\cdot 4} + c_0(-1)^{-3\cdot 4}F_{3\cdot 5}\right) \\
c_4 &= -\frac{1}{2}\left(-4F_6 + 3F_9 + 186F_{12} - 44F_{15}\right) \\
c_4 &= -\frac{1}{2}\left(-4\cdot 8 + 3\cdot 34 + 186\cdot 144 - 44\cdot 610\right) = -7 \quad \textit{and} \\
c_n &= \frac{1}{F_3}\left(c_3(-1)^{-3\cdot 1}F_{3\cdot 1} + c_2(-1)^{-3\cdot 2}F_{3\cdot 2} + c_1(-1)^{-3\cdot 3}F_{3\cdot 3} + c_0(-1)^{-3\cdot 4}F_{3\cdot 4}\right) \\
c_5 &= \frac{1}{2}\left(4(-1)^3F_3 + 3(-1)^6F_6 + (-186)(-1)^9F_9 + (-44)(-1)^{12}F_{12}\right) \\
c_5 &= \frac{1}{2}\left(-4\cdot 2 + 3\cdot 8 + 186\cdot 34 - 44\cdot 144\right) = 2.
\end{aligned}$$

Example 3.5. *$P_5(x) = 12\frac{5}{16}x^5 - 103\frac{7}{8}x^4 - 4x^3 + 3x^2 - 5x + 2$ may be confirmed to have zeros $2\alpha^3$ and $2\beta^3$ using formulas* (2.2) *and* (3.6) *to match c_1 and c_0 , or* (2.15) *and* (3.12) *to match c_4 and c_5.*

The following example illustrates a method of checking whether a zero of a polynomial is a duplicate zero.

Example 3.6. *Is* 3 *a duplicate zero of the polynomial*

$$-46530 + 18033x + 5x^2 + 6x^3 - 3x^4 - x^5 + 2x^6 - 4x^7?$$

Using equations (2.2) and (2.3) with $r = 3$:

$$\begin{aligned} c_1 &= -\left(2c_2 \cdot 3 + 3c_3 \cdot 3^2 + 4c_4 \cdot 3^3 + 5c_5 \cdot 3^4 + 6c_6 \cdot 3^5 + 7c_7 \cdot 3^6\right) \\ &= -\left(2 \cdot 5 \cdot 3 + 3 \cdot 6 \cdot 9 + 4(-3) \cdot 27 + 5(-1) \cdot 81 + 6 \cdot 2 \cdot 243 + 7(-4) \cdot 729\right) = 18033, \quad \textit{and} \\ c_0 &= c_2 \cdot 3^2 + 2c_3 \cdot 3^3 + 3c_4 \cdot 3^4 + 4c_5 \cdot 3^5 + 5c_6 \cdot 3^6 + 6c_7 \cdot 3^7 \\ &= 5 \cdot 9 + 2 \cdot 6 \cdot 27 + 3(-3) \cdot 81 + 4(-1) \cdot 243 + 5 \cdot 2 \cdot 729 + 6(-4) \cdot 2187 = -46530 \end{aligned}$$

Since these are the correct values of c_1 and c_0, we may conclude that there is a duplicate zero at $x = r = 3$. This is confirmed by an alternate method for this, checking if the number is a zero of both the polynomial and its derivative. The graph of the polynomial shows it to have a relative maximum at the point $(3, 0)$.

Example 3.7. *Check if $2 \pm \sqrt{7}$ are exact roots of the cubic equation $5x^3 - 42x^2 + 73x + 66 = 0$, and if so, what is the third root?*

$p = a + b = (2 + \sqrt{7}) + (2 - \sqrt{7}) = 4$, $q = ab = 2^2 - \sqrt{7}^2 = -3$. *For the U_n sequence, $U_1 = 1$, and $U_2 = p$. To find U_3, we use $U_n = pU_{n-1} - qU_{n-2}$, $U_3 = p^2 - q = 16 - (-3) = 19$. $r = 1$, and $k = 1$, therefore we may use formulas (3.11)(A) and (B). $c_1 = -\left(c_2U_2 + c_3U_3\right) = -\left(-42 \cdot 4 + 5 \cdot 19\right) = 73$. $c_0 = q\left(c_2U_1 + c_3U_2\right) = -3\left(-42 \cdot 1 + 5 \cdot 4\right) = 66$. These values for c_1 and c_0 match those in the equation, therefore $2 \pm \sqrt{7}$ are exact roots of the equation. If two of the roots of a cubic equation are also roots of $q - px + x^2 = 0$, the third cubic root is $-(\frac{c_2}{c_3} + p)$. Hence the third root here is $-(\frac{-42}{5} + 4) = \frac{22}{5}$.*

4. Application of Proposition 2.1 in the Construction of Identities

Many identities of F_n, L_n, and the golden ratio can be generated from Proposition 2.1 and its corollaries. We will use the generalized formulas so that the identities will apply to any quadratic root pair, a and b, with associated numbers sequences, U_n and V_n, as described by Lucas in [10].

The identities in the following two corollaries are derived from the particular polynomials in which $r = 1$, $c_n = 1$ and $c_i = 0$ for $2 \le i \le n - 1$.

Corollary 4.1. *Suppose that $P_n(x)$, defined as (2.1), has zeros a^k and b^k, and its coefficients $c_n = 1$ and $c_i = 0$ for $2 \le i \le n - 1$, then*

$$\text{(A)}\ U_k a^{kn} = U_{kn} a^k - q^k U_{k(n-1)}, \qquad \text{(B)}\ U_k b^{kn} = U_{kn} b^k - q^k U_{k(n-1)}, \tag{4.1}$$

$$U_k V_{kn} = U_{kn} V_k - 2q^k U_{k(n-1)}, \tag{4.2}$$

$$\text{(A)}\ U_{k(n-1)} a^{kn} = U_{kn} a^{k(n-1)} - q^{k(n-1)} U_k, \qquad \text{(B)}\ U_{k(n-1)} b^{kn} = U_{kn} b^{k(n-1)} - q^{k(n-1)} U_k, \tag{4.3}$$

$$U_{k(n-1)} V_{kn} = U_{kn} V_{k(n-1)} - 2q^{k(n-1)} U_k, \tag{4.4}$$

$$\text{(A)} \quad U_m a^n = U_n a^m - q^m U_{n-m}, \qquad \text{(B)} \quad U_m b^n = U_n b^m - q^m U_{n-m}, \tag{4.5}$$

$$U_m V_n = U_n V_m - 2q^m U_{n-m}, \tag{4.6}$$

$$U_{m+n} = U_m a^n + U_n b^m, \ \text{ and} \tag{4.7}$$

$$2U_{m+n} = U_m V_n + U_n V_m \,. \tag{4.8}$$

Proof. Since $P_n(a^k) = P_n(b^k) = 0$, noting $c_n = 1$ and $c_i = 0$ for $2 \le i \le n-1$ in $P_n(x)$, we have $P_n(a^k) = c_0 + c_1 a^k + a^{kn}$. Thus, $a^{kn} = -c_1 a^k - c_0$. Substituting the values for c_1 and c_0 given by equations (2.11) and (3.8) into the previous equation results in

$$a^{kn} = \frac{U_{kn}}{U_k} a^k - \frac{q^k U_{k(n-1)}}{U_k} = \frac{U_{kn}}{U_k} a^k - \frac{q^k U_{k(n-1)}}{U_k}.$$

Multiplying by U_k, we get $U_k a^{kn} = U_{kn} a^k - q^k U_{k(n-1)}$, which is equation (4.1)(A). In a similar manner, one may obtain (4.1)(B).

Adding equations (4.1)(A) and (B) we get

$$U_k \left(a^{kn} + b^{kn}\right) = U_{kn} \left(a^k + b^k\right) - 2q^k U_{k(n-1)},$$

which can be simplified to $U_k V_{kn} = U_{kn} V_k - 2q^k U_{k(n-1)}$, i.e., equation (4.2). Equating the RHS of equation (2.6) to the RHS of (3.6)(C) (using $i = n$ only), and simplifying the resulting equation will also yield identity (4.2).

To prove identity (4.3)(A), Identity (4.1)(B) may be written as the polynomial equation:

$$U_k b^{kn} - U_{kn} b^k + q^k U_{k(n-1)} = 0.$$

From lemma 2.5, and noting $b^{-k} = \left(q^{-1}a\right)^k$, we know that

$$q^k U_{k(n-1)} \left(q^{-1}a\right)^{kn} - U_{kn} \left(q^{-1}a\right)^{k(n-1)} + U_k = 0,$$

which implies

$$q^{-k(n-1)} U_{k(n-1)} a^{kn} = q^{-k(n-1)} U_{kn} a^{k(n-1)} - U_k.$$

Hence,

$$U_{k(n-1)} a^{kn} = U_{kn} a^{k(n-1)} - q^{k(n-1)} U_k,$$

A similar argument can be applied to identity (4.1)(A) of Corollary 4.1 for generating identity (4.3)(B). Adding identities (4.3)(A) and (B) produces identity (4.4).

If the variable k in the identities (4.1)(A) and (B), and (4.2) is replaced with m, and kn is replaced with $\mathbf{n}$, these identities become equivalent to identities (4.5)(A) and (B), and (4.6). Another way to prove these identities is to replace the U_n, U_m, and U_{n-m} with their respective Lucas Binet or Lucas formula representations in equations (4.5)(A) and (B). Then using $b^m = q^m a^{-m}$ in (4.5)(A), and $a^m = q^m b^{-m}$ in (4.5)(B), and simplifying, it is easily seen that these equations are valid. Adding equations (4.5)(A) and (4.5)(B) produces identity (4.6).

Applying $U_{-m} = -q^{-m} U_m$ and $a^{-1} = q^{-1} b$ to identity (4.5)(A), then dividing by $-q^{-m}$, converts it into identity (4.7). Equation (4.8) is the result of adding identity(4.7) to itself after interchanging the subscripts.

□

When $k = 1$, $a = \alpha$, and $b = \beta$, the identities (4.1)(A) and (B) reduce to $\alpha^n = \alpha F_n + F_{n-1}$ and $\beta^n = \beta F_n + F_{n-1}$, which are the identities proved by Thoro [17] mentioned in the introduction. Also, identity (4.2) reduces to $L_n = F_n + 2F_{n-1}$, which is identity E [8, p. 27].

When $k = 1$, $a = \alpha$, and $b = \beta$, identities (4.3)(A) and (B), and (4.4) become:

$$\text{(A)} F_{n-1}\alpha^n = F_n \alpha^{n-1} + (-1)^n \quad \text{(B)} F_{n-1}\beta^n = F_n \beta^{n-1} + (-1)^n \quad \text{(C)} F_{n-1} L_n = F_n L_{n-1} + 2(-1)^n \tag{4.9}$$

Compare the single result of the identity mentioned in the introduction, $\alpha^n = \alpha F_n + F_{n-1}$, with $n = 6$, $\alpha^6 = 8\alpha + 5$, to the multiple results using identity (4.5)(A), which provides all of the following identities for α^6, and more.

$$\begin{array}{lll} \alpha^6 = 8\alpha + 5 & \alpha^6 = 8\alpha^2 - 3 & \alpha^6 = 4\alpha^3 + 1 \\ \alpha^6 = \dfrac{1}{3}\left(8\alpha^4 - 1\right) & \alpha^6 = \dfrac{1}{5}\left(8\alpha^5 + 1\right) & \alpha^6 = \dfrac{1}{13}\left(8\alpha^7 + 1\right) \\ \alpha^6 = \dfrac{1}{21}\left(8\alpha^8 - 1\right) & \alpha^6 = \dfrac{1}{34}\left(8\alpha^9 + 2\right) & \alpha^6 = \dfrac{1}{55}\left(8\alpha^{10} - 3\right) \end{array}$$

Remark The F_n, L_n version of identity (4.8) was proposed by Ferns in [6], and proved by Wall in [20], and is also listed in [15] as identity (5).

Corollary 4.2. *Let $P_n(x) = \sum_{i=0}^{n} c_i x^i$ have zeros a^k and b^k, and its coefficients $c_n = 1$ and $c_i = 0$ for $2 \le i \le n-1$, then*

$$\text{(A) } U_k a^{kn} = U_{kn}\left(a^k - \frac{V_k}{2}\right) + \frac{U_k V_{kn}}{2} \qquad \text{(B) } U_k b^{kn} = U_{kn}\left(b^k - \frac{V_k}{2}\right) + \frac{U_k V_{kn}}{2} \tag{4.10}$$

Proof. Because $P_n(a^k) = c_0 + c_1 a^k + a^{kn}$, one may have $a^{kn} = -c_1 a^k - c_0 = 0$. Similarly, $b^{kn} = -c_1 b^k - c_0$. Substituting the values for c_1 and c_0 given by equations (2.11) and (2.6) into these equations results in the following

$$a^{kn} = \frac{U_{kn}}{U_k} a^k + \frac{V_{kn}}{2} - \frac{V_k U_{kn}}{2U_k}.$$

Multiplying both sides by U_k produces

$$U_k a^{kn} = U_{kn}\left(a^k - \frac{V_k}{2}\right) + \frac{U_k V_{kn}}{2}.$$

Similarly, we have

$$U_k b^{kn} = U_{kn}\left(b^k - \frac{V_k}{2}\right) + \frac{U_k V_{kn}}{2}.$$

□

When $k = 1$, $a = \alpha$, and $b = \beta$, the identities (4.10)(A) and (B) reduce to: $\alpha^n = F_n\left(\alpha - \frac{1}{2}\right) + \frac{L_n}{2}$, and $\beta^n = F_n\left(\beta - \frac{1}{2}\right) + \frac{L_n}{2}$, which may also be derived from the identities $\alpha^n = \alpha F_n + F_{n-1}$, and $L_n = F_n + 2F_{n-1}$. It is obvious that the case of $k = 0$ is trivial and is omitted from Corollaries 3.2 and 3.3.

The following three corollaries are for the particular polynomials $P_n(x)$, defined by (2.1), with $r = 1$, and coefficients $c_i = t$, $t \ne 0$, for $2 \le i \le n$.

Corollary 4.3. *Let $P_n(x) = \sum_{i=0}^{n} c_i x^i$ have zeros a and b, and $c_i = t$, $t \ne 0$, for $2 \le i \le n$, then*
$c_1 = t\left(1 - \frac{1}{p-q-1}\left[U_{n+2} - 1 - (p-1)U_{n+1}\right]\right)$ *and*
$c_0 = t\left(U_n - \frac{1}{p-q-1}\left[U_{n+2} - 1 - (p-1)U_{n+1}\right]\right)$. *For $a = \alpha$, and $b = \beta$ this is $c_1 = -t(F_{n+2} - 2)$ and $c_0 = -t(F_{n+1} - 1)$.*

$U_0 =$	$-1/q\,U_2$	$+$	$p/q\,U_1$
$U_1 =$	$-1/q\,U_3$	$+$	$p/q\,U_2$
$U_2 =$	$-1/q\,U_4$	$+$	$p/q\,U_3$
.	.		.
.	.		.
$U_{n-2} =$	$-1/q\,U_n$	$+$	$p/q\,U_{n-1}$
$U_{n-1} =$	$-1/q\,U_{n+1}$	$+$	$p/q\,U_n$
$U_n =$	$-1/q\,U_{n+2}$	$+$	$p/q\,U_{n+1}$
$\sum_{i=0}^{n} U_i =$	$-\frac{1}{q}\left(\sum_{i=1}^{n} U_i + U_{n+1} + U_{n+2} - U_1\right)$	$+$	$\frac{p}{q}\left(\sum_{i=1}^{n} U_i + U_{n+1}\right)$

Table 2

Proof. With $c_i = t$ for $2 \le i \le n$, equations (3.11)(A) and (B) become

$$c_1 = -t\sum_{i=2}^{n} U_i, \quad \textit{and} \quad c_0 = tq\sum_{i=2}^{n} U_{i-1}.$$

Rearranging the formula $U_{n+2} = pU_{n+1} - qUn$, [10, eqns. (10), p. 10], we list the consecutive U_i of $\sum_{i=0}^{n} U_i$ in tabular form for summation. Observing Table 2, and noting that $U_0 = 0$ and $U_1 = 1$, it can been seen that

$$\sum_{i=1}^{n} U_i = \frac{1}{p-q-1}\left(U_{n+2} - 1 - (p-1)U_{n+1}\right) \tag{4.11}$$

When $p = 1$, and $q = -1$, equation (4.11) becomes the identity $\sum_{i=1}^{n} F_i = F_{n+2} - 1$, [10, p. 26], and [8, p. 52] I_1.

$c_1 = -t\left(\sum_{i=1}^{n} U_i - U_1\right)$, with identity (4.11), this is $t\left(1 - \frac{1}{p-q-1}\left[U_{n+2} - 1 - (p-1)U_{n+1}\right]\right)$. When $p = 1$, and $q = -1$, this is $c_1 = -t(F_{n+2} - 2)$. Next, Let $m = i - 1$, then when $i = 2$, $m = 1$, and $i = n$, we have $m = n - 1$. Then $c_0 = -t\sum_{i=2}^{n} U_{i-1}$ is transformed into $c_0 = t\left(U_n - \sum_{m=1}^{n} U_m\right)$, and with identity (4.11), this becomes $t\left(U_n - \frac{1}{p-q-1}\left[U_{n+2} - 1 - (p-1)U_{n+1}\right]\right)$. When $p = 1$, and $q = -1$, this is $c_0 = t\left(F_n - F_{n+2} + 1\right)$. Replacing F_{n+2} with $F_n + F_{n+1}$ results in $c_0 = t\left[F_n - (F_n + F_{n+1}) + 1\right] = -t(F_{n+1} - 1)$.

□

The following identities are generated from Corollary 4.3.

Corollary 4.4. *Suppose that $P_n(x)$, defined as (2.1), has zeros α and β, and coefficients $c_i = 1$ for $2 \le i \le n$, then*

$$\text{(A)}\ \sum_{i=0}^{n} \alpha^i = \alpha(F_{n+2} - 1) + F_{n+1} \quad \text{(B)}\ \sum_{i=0}^{n} \alpha^i = \alpha^{n+2} - \alpha \tag{4.12}$$

$$\text{(A)}\ \sum_{i=0}^{n} \beta^i = \beta(F_{n+2} - 1) + F_{n+1} \quad \text{(B)}\ \sum_{i=0}^{n} \beta^i = \beta^{n+2} - \beta \tag{4.13}$$

Proof. Substituting $-(F_{n+2}-2)$ for c_1 and $-(F_{n+1}-1)$ for c_0 into $P_n(\alpha) = c_0 + c_1\alpha + \sum_{i=2}^{n}(1)\alpha^i = 0$ produces

$$P_n(\alpha) = -(F_{n+1}-1) - (F_{n+2}-2)\alpha + \sum_{i=2}^{n}\alpha^i = 0, \tag{4.14}$$

which implies

$$1 + \alpha + \sum_{i=2}^{n}\alpha^i = 1 + \alpha + \alpha(F_{n+2}-2) + F_{n+1} - 1$$

or equivalently,

$$\sum_{i=0}^{n}\alpha^i = \alpha(F_{n+2}-1) + F_{n+1},$$

i.e., (4.12)(A). Replacing $\alpha F_{n+2} + F_{n+1}$ with α^{n+2} (Thoro identity):

$$\sum_{i=0}^{n}\alpha^i = \alpha^{n+2} - \alpha,$$

which is (4.12)(B). Doing the same process to

$$P_n(\beta) = c_0 + c_1\beta + \sum_{i=2}^{n}(1)\beta^i = 0$$

produces identity (4.13)(A). Replacing $\beta F_{n+2} + F_{n+1}$ with β^{n+2} (see Thoro's identity), we have

$$\sum_{i=1}^{n}\beta^i = \beta^{n+2} - \beta - 1,$$

which is (4.13)(B).

□

Adding identities (4.12)(B) and (4.13)(B) gives us $\sum_{i=1}^{n} L_i = L_{n+2} - 3$ which is identity (I_2) in [8, p. 54]. Adding identities (4.12)(A) and (4.13)(A) ends up with that same result when the identity $L_{n+2} = F_{n+2} + 2F_{n+1}$ is applied.

Another way to prove identities (4.12)(A) and (4.13)(A) is to use the identities

$$\alpha^n = \alpha F_n + F_{n-1} \quad \textit{and} \quad \beta^n = \beta F_n + F_{n-1},$$

with $\sum_{i=1}^{n} F_i = F_{n+2} - 1$. More precisely,

$$\sum_{i=1}^{n}\alpha^i = \alpha\sum_{i=1}^{n}F_i + \sum_{i=1}^{n}F_{i-1} = \alpha(F_{n+2}-1) + \sum_{m=0}^{n-1}F_m = \alpha(F_{n+2}-1) + F_{n+1} - 1,$$

which implies

$$1 + \sum_{i=1}^{n}\alpha^i = 1 + \alpha(F_{n+2}-1) + F_{n+1} - 1,$$

or equivalently,

$$\sum_{i=0}^{n}\alpha^i = \alpha(F_{n+2}-1) + F_{n+1}.$$

A different proof of identities (4.12)(B) and (4.13)(B) is to use the formula for the sum of a geometric progression having α (or β) as both the first term and the common ratio. For example, Let

$$s_n := \sum_{i=1}^{n} \alpha^i = \alpha \frac{\alpha^n - 1}{\alpha - 1}.$$

Since $\alpha - 1 = \frac{1}{\alpha}$, $s_n = \alpha^2 (\alpha^n - 1) = \alpha^{n+2} - \alpha^2$. Replacing α^2 with $\alpha + 1$, $s_n = \alpha^{n+2} - (\alpha + 1)$, which gives

$$1 + \sum_{i=1}^{n} \alpha^i = 1 + \alpha^{n+2} - \alpha - 1,$$

namely, $\sum_{i=0}^{n} \alpha^i = \alpha^{n+2} - \alpha$.

Corollary 4.5. *If $P_n(x) = \sum_{i=0}^{n} c_i x^i$ has 2α and 2β as zeros, and coefficients $c_i = 1$ for $2 \le i \le n$, then coefficients*

$$c_1 = -\frac{2^n L_{n+1} - 6}{5} \quad \textit{and} \quad c_0 = -\frac{2^{n+1} L_n - 4}{5}.$$

Proof. Using Proposition 2.1, $r = 2$, and $k = 1$, then

$$c_1 = -\sum_{i=2}^{n} 2^{i-1} F_i \quad \textit{and} \quad c_0 = -\sum_{i=2}^{n} 2^i F_{i-1}.$$

Hence, to prove the formula for c_1, we need to verify that

$$\sum_{i=2}^{n} 2^{i-1} F_i = \frac{2^n L_{n+1} - 6}{5}.$$

First, it is clearly true for $n = 2$, because both sides are equal to 2. Assume that it is true for n. To check if the formula for the sum is true for $n+1$, we replace n with $n+1$ in the sum:

$$\sum_{i=2}^{n+1} 2^{i-1} F_i = \sum_{i=2}^{n} 2^{i-1} F_i + 2^n F_{n+1} = \frac{2^n L_{n+1} - 6}{5} + 2^n F_{n+1}.$$

Using the identity $L_{n-1} + L_{n+1} = 5F_n$, from [18, identity (5), p. 176], we replace F_{n+1} by $\frac{L_n + L_{n+2}}{5}$ and obtain

$$\sum_{i=2}^{n+1} 2^{i-1} F_i = \frac{2^n L_{n+1} - 6}{5} + 2^n \left(\frac{L_n + L_{n+2}}{5} \right) = \frac{2^{n+1} L_{n+2} - 6}{5},$$

which matches the original formula when $n = n + 1$. Therefore, if it is true for n, then it is true for $n + 1$. Since it is true for $n = 2$, by mathematical induction it is true for all integers $n \ge 2$, which completes the proof of the formula for c_1.

To prove the formula for c_0, we need to show

$$\sum_{i=2}^{n} 2^i F_{i-1} = \frac{2^{n+1} L_n - 4}{5}.$$

It is easily seen that the above equation holds for $n = 2$ because both sides becomes 4. We note that for $n > 2$,

$$\sum_{i=2}^{n} 2^i F_{i-1} = 4 \left(1 + \sum_{i=2}^{n-1} 2^{i-1} F_i \right),$$

in which we substitute the identity

$$\sum_{i=2}^{n-1} 2^{i-1} F_i = \frac{2^{n-1} L_n - 6}{5}$$

and obtain

$$\sum_{i=2}^{n} 2^i F_{i-1} = 4\left(1 + \frac{2^{n-1} L_n - 6}{5}\right) = \frac{2^{n+1} L_n - 4}{5}.$$

□

Remark The c_0 values of $P_n(x) = 0 \,|\, P_n(2\alpha) = P_n(2\beta) = 0$ when $\{c_2, \ldots, c_n\} = 1$ are $-4a_n$, where a_n are the integers of Sloane's sequence A014335 [11] starting with $a_1 = 0$ and $a_2 = 1$. The identities proven in Corollary 4.5 conform to the two formulas attributed to Benoit Cloitre on the web page for this sequence [11]. It is also $-2b_n$, where $b_n \in$ Sloane's sequence A014334, [12].

The next two corollaries are for the particular polynomials in which $c_i = t(-1)^{n-i}$, $t \neq 0$, for $2 \leq i \leq n$.

Corollary 4.6. *Let $P_n(x) = \sum_{i=0}^{n} c_i x^i$ have zeros α and β, and coefficients $c_i = t(-1)^{n-i}$ for $2 \leq i \leq n$, then $c_1 = -tF_{n-1}$ and $c_0 = -t(F_{n-2} + (-1)^n)$.*

Proof. [8, p. 56] presents the identities

$$F_1 + F_3 + F_5 + \ldots + F_{2m-1} = F_{2m}\,, \qquad for \;\; m \geq 1 \tag{I_5}$$

$$F_2 + F_4 + F_6 + \ldots + F_{2m} \;=\; F_{2m+1} - 1\,, \qquad for \;\; m \geq 1 \tag{I_6}$$

For the purpose of this proof, These identities are combined into one, and the F_1 term is omitted. If m in (I_5) is replaced with $\frac{n+1}{2}$, $F_1 = 1$ is subtracted from (I_5), m in (I_6) is replaced with $\frac{n}{2}$, and we introduce the parity function

$$\epsilon = \frac{1 - (-1)^n}{2} = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

we get the following identity from (I_5) and (I_6):

$$\sum_{j=0}^{2j=n-2-\epsilon} F_{n-2j} = F_n + F_{n-2} + F_{n-4} + \ldots + F_{2+\epsilon} = F_{n+1} - 1\,, \qquad n \geq 2 \tag{4.15}$$

This identity is for $2 \leq n - 2j \leq n$ (because the F_1 term is omitted). For $1 \leq n - 2j \leq n$ (in which the F_1 term is included when n is odd), the identity is:

$$\sum_{j=0}^{2j=n-2+\epsilon} F_{n-2j} = F_n + F_{n-2} + F_{n-4} + \ldots + F_{2-\epsilon} = F_{n+1} - 1 + \epsilon\,, \qquad n \geq 1 \tag{4.16}$$

Letting $c_i = t(-1)^{n-i}$ for $2 \leq i \leq n$, equation (3.11)(A) becomes

$$c_1 = t\left(-F_n + F_{n-1} - F_{n-2} + F_{n-3} - \ldots + (-1)^n F_3 - (-1)^n F_2\right).$$

The above equation may be separated into

$$c_1 = -t\left(F_n + F_{n-2} + F_{n-4} + \ldots + F_{2+\epsilon}\right) + t\left(F_{n-1} + F_{n-3} + F_{n-5} + \ldots + F_{3-\epsilon}\right).$$

Using identity (4.15), we convert c_1 to

$$c_1 = -t\,(F_{n+1} - 1) + t\,(F_{n-1+1} - 1) = -tF_{n+1} + tF_n = -tF_{n-1}.$$

Inserting $c_i = t(-1)^{n-i}$ for $2 \le i \le n$ into equation (3.11)(B), we have

$$c_0 = t\,(-F_{n-1} + F_{n-2} - F_{n-3} + F_{n-4} - \ldots + (-1)^n F_2 - (-1)^n F_1)\,.$$

Applying identity(4.15), the above equation for c_0 is converted to

$$c_0 = -t\,(F_{n-1+1} - 1) + t\,(F_{n-2+1} - 1) - t(-1)^n = -tF_n + tF_{n-1} - t(-1)^n = -t(F_{n-2} + (-1)^n).$$

□

Corollary 4.6 is used to derive the following identities.

Corollary 4.7. *Suppose that $P_n(x)$, defined as (2.1), has zeros α and β, and $c_i = (-1)^{n-i}$ for $2 \le i \le n$, then*

$$\text{(A)}\ \sum_{i=1}^{n} (-1)^{n-i}\alpha^i = \alpha\,(F_{n-1} - (-1)^n) + F_{n-2} + (-1)^n$$
$$\text{(B)}\ \sum_{i=1}^{n} (-1)^{n-i}\alpha^i = \alpha^{n-1} + (-1)^n\,(1 - \alpha) \tag{4.17}$$

$$\text{(A)}\ \sum_{i=1}^{n} (-1)^{n-i}\beta^i = \beta\,(F_{n-1} - (-1)^n) + F_{n-2} + (-1)^n$$
$$\text{(B)}\ \sum_{i=1}^{n} (-1)^{n-i}\beta^i = \beta^{n-1} + (-1)^n\,(1 - \beta) \tag{4.18}$$

$$\sum_{i=1}^{n} (-1)^{n-i} L_i = L_{n-1} + (-1)^n \tag{4.19}$$
$$\sum_{i=1}^{n} (-1)^{n-i} F_i = F_{n-1} - (-1)^n. \tag{4.20}$$

Proof. Substituting $-F_{n-1}$ for c_1 and $-(F_{n-2} + (-1)^n)$ for c_0 into

$$P_n(\alpha) = c_0 + c_1\alpha + \sum_{i=2}^{n} (-1)^{n-i}\alpha^i = 0$$

produces

$$P_n(\alpha) = -(F_{n-2} + (-1)^n) - \alpha F_{n-1} + \sum_{i=2}^{n} (-1)^{n-i}\alpha^i = 0,$$

which implies

$$\sum_{i=2}^{n} (-1)^{n-i}\alpha^i = \alpha F_{n-1} + F_{n-2} + (-1)^n.$$

Hence,

$$\sum_{i=1}^{n}(-1)^{n-i}\alpha^i = -(-1)^n\alpha + \sum_{i=2}^{n}(-1)^{n-i}\alpha^i = \alpha\left(F_{n-1} - (-1)^n\right) + F_{n-2} + (-1)^n,$$

which is (4.17)(A). Replacing $\alpha F_{n-1} + F_{n-2}$ with α^{n-1}, from the above formula we have

$$\sum_{i=1}^{n}(-1)^{n-i}\alpha^i = \alpha^{n-1} + (-1)^n\left(1-\alpha\right),$$

which is (4.17)(B).

Substituting $-F_{n-1}$ for c_1 and $-(F_{n-2} + (-1)^n)$ for c_0 into

$$P_n(\beta) = c_0 + c_1\beta + \sum_{i=2}^{n}(-1)^{n-i}\beta^i = 0$$

produces

$$P_n(\beta) = -(F_{n-2} + (-1)^n) - \beta F_{n-1} + \sum_{i=2}^{n}(-1)^{n-i}\beta^i = 0$$

which implies

$$\sum_{i=2}^{n}(-1)^{n-i}\beta^i = \beta F_{n-1} + (F_{n-2} + (-1)^n).$$

Hence,

$$\sum_{i=1}^{n}(-1)^{n-i}\beta^i = (-1)^{n-1}\beta + \sum_{i=2}^{n}(-1)^{n-i}\beta^i = \beta\left(F_{n-1} - (-1)^n\right) + F_{n-2} + (-1)^n,$$

which is (4.18)(A). Replacing $\beta F_{n-1} + F_{n-2}$ with β^{n-1} into the above formula yields

$$\sum_{i=1}^{n}(-1)^{n-i}\beta^i = \beta^{n-1} + (-1)^n\left(1-\beta\right),$$

which is (4.18)(B).

Adding identities (4.17)(A) and (4.18)(A) gives us

$$\sum_{i=1}^{n}(-1)^{n-i}\left(\alpha^i + \beta^i\right) = (\alpha+\beta)\left(F_{n-1} - (-1)^n\right) + 2F_{n-2} + 2(-1)^n,$$

which implies

$$\sum_{i=1}^{n}(-1)^{n-i}L_i = (1)\left(F_{n-1} - (-1)^n\right) + 2F_{n-2} + 2(-1)^n.$$

Replacing $F_{n-1} + 2F_{n-2}$ with L_{n-1} yields

$$\sum_{i=1}^{n}(-1)^{n-i}L_i = L_{n-1} + (-1)^n,$$

which is (4.19).

Subtracting identities (4.17)(A) and (4.18)(A) gives us

$$\sum_{i=1}^{n}(-1)^{n-i}\left(\alpha^i-\beta^i\right)=(\alpha-\beta)\left(F_{n-1}-(-1)^n\right).$$

Dividing by $\alpha-\beta$ on the both sides of the above equation, we obtain

$$\sum_{i=1}^{n}(-1)^{n-i}\left(\frac{\alpha^i-\beta^i}{\alpha-\beta}\right)=F_{n-1}-(-1)^n,$$

or equivalently,

$$\sum_{i=1}^{n}(-1)^{n-i}F_i=F_{n-1}-(-1)^n,$$

which is (4.20).

□

Corollary 4.8. *If* $P_n(\alpha)=P_n(\beta)=0$, *and* $c_i=t(-1)^i$, $t\neq 0$, *for* $2\leq i\leq n$, *then* $c_1=-t(-1)^nF_{n-1}$ *and* $c_0=-t\left[(-1)^nF_{n-2}+1\right]$.

Proof. By equations (3.11) (A) and (B), $c_1=-t\sum_{i=2}^{n}(-1)^iF_i$, and $c_0=-t\sum_{i=2}^{n}(-1)^iF_{i-1}$.
$c_1=-t\left[(-1)^nF_n+(-1)^{n-1}F_{n-1}+\ldots-F_3+F_2\right]=-t(-1)^n\left[F_n-F_{n-1}+\ldots-F_3+F_2\right]$.
By identity (4.15), $c_1=-t(-1)^n\left[(F_{n+1}-1)-(F_n-1)\right]=-t(-1)^nF_{n-1}$.
$c_0=-t\left[(-1)^nF_{n-1}+(-1)^{n-1}F_{n-2}+\ldots+F_3-F_2+F_1\right]=$
$-t(-1)^n\left[F_{n-1}-F_{n-2}+\ldots-F_2+F_1\right]$. By identity (4.15),
$c_0=-t(-1)^n\left[(F_n-1)-(F_{n-1}-1)+F_1\right]=-t\left[(-1)^nF_{n-2}+1\right]$. □

Corollary 4.8 is used to derive the following identities.

Corollary 4.9.

$$\text{(A)}\ \sum_{i=1}^{n}(-\alpha)^i=(-1)^n(\alpha F_{n-1}+F_{n-2})+\beta\quad \text{(B)}\ \sum_{i=1}^{n}(-\alpha)^i=(-\alpha)^{n-1}+\beta \tag{4.21}$$

$$\text{(A)}\ \sum_{i=1}^{n}(-\beta)^i=(-1)^n(\beta F_{n-1}+F_{n-2})+\alpha\quad \text{(B)}\ \sum_{i=1}^{n}(-\beta)^i=(-\beta)^{n-1}+\alpha \tag{4.22}$$

$$\sum_{i=1}^{n}(-1)^iL_i=(-1)^nL_{n-1}+1 \tag{4.23}$$

$$\sum_{i=1}^{n}(-1)^iF_i=(-1)^nF_{n-1}-1 \tag{4.24}$$

Proof. Let $t=1$, and substitute the values from corollary 4.8 for c_1 and c_0 into
$P_n(\alpha)=c_0+c_1\alpha+\sum_{i=2}^{n}(-1)^i\alpha^i=0$ to get $\sum_{i=2}^{n}(-1)^i\alpha^i=(-1)^nF_{n-1}\alpha+(-1)^nF_{n-2}+1$.
Adding $(-1)^1\alpha^1=-\alpha$ to both sides, and replacing $1-\alpha$ with β:
$\sum_{i=1}^{n}(-\alpha)^i=(-1)^n(\alpha F_{n-1}+F_{n-2})+\beta$, which is (4.21)(A). Replacing $\alpha F_{n-1}+F_{n-2}$ with α^{n-1}: $\sum_{i=1}^{n}(-\alpha)^i=(-1)^n\alpha^{n-1}+\beta$, which is (4.21)(B).
This same process may be done to $P_n(\beta)=c_0+c_1\beta+\sum_{i=2}^{n}(-1)^i\beta^i=0$ to arrive at identities (4.22) (A) and (B).

Adding equations (4.21) (B) and (4.22) (B) to get (4.23):
$\sum_{i=1}^{n}(-1)^i(\alpha^i+\beta^i)=(-1)^n\left(\alpha^{n-1}+\beta^{n-1}\right)+\alpha+\beta \to \sum_{i=1}^{n}(-1)^i L_i=(-1)^n L_{n-1}+1$, which is (4.23). Subtracting equation (4.22) from equation (4.21) to get (4.24):
$\sum_{i=1}^{n}(-1)^i(\alpha^i-\beta^i)=(-1)^n\left(\alpha^{n-1}-\beta^{n-1}\right)-(\alpha-\beta) \to$ Divide by $(\alpha-\beta)$ $\to$
$\sum_{i=1}^{n}(-1)^i\frac{\alpha^i-\beta^i}{\alpha-\beta}=(-1)^n\left(\frac{\alpha^{n-1}+\beta^{n-1}}{\alpha-\beta}\right)-\frac{\alpha-\beta}{\alpha-\beta} \to \sum_{i=1}^{n}(-1)^i F_i=(-1)^n F_{n-1}-1$, which is (4.24). $\square$

Corollary 4.10. *Let $P_n(x)=\sum_{i=0}^{n}c_ix^i$ have r as a duplicate zero, and its coefficients $c_i=t_1$ for $2\le i\le n$, then*

$$c_1=-t_1\sum_{i=2}^{n} i r^{i-1}=\begin{cases}-t_1\left(\frac{2r-r^2-(n+1)r^n+nr^{n+1}}{(r-1)^2}\right) & if\ r\neq 1\\ -t_1\left(\frac{n(n+1)}{2}-1\right) & if\ r=1.\end{cases} \tag{4.25}$$

$$c_0=t_1\sum_{i=2}^{n}(i-1)r^i=\begin{cases}t_1\left(\frac{r^2-nr^{n+1}+(n-1)r^{n+2}}{(r-1)^2}\right) & if\ r\neq 1\\ t_1\left(\frac{n(n-1)}{2}\right) & if\ r=1.\end{cases}. \tag{4.26}$$

Proof. To proof the case of $r=1$, using the equations of the $k=0$ case of (2.2) and (2.3), with $r=1$ and $c_2,\ldots,c_n=t_1$, we have

$$c_1=-\sum_{i=2}^{n}t_1 i \quad \text{and} \quad c_0=\sum_{i=2}^{n}t_1(i-1)$$

From these, and Gauss's formula, $\sum_{i=1}^{n} i=\frac{n(n+1)}{2}$, we get $c_1=-t_1\left(\frac{n(n+1)}{2}-1\right)$. Letting $m=i-1$, we have $c_0=t_1\sum_{m=1}^{n-1}m=t_1\left(\frac{n(n-1)}{2}\right)$.

To prove equations (4.25) and (4.26) for the case $r\neq 1$, we divide out the $\pm t_1$, and label the sum on the left as S, then multiply both sides of the equation by $(1-r)^2=1-2r+r^2$ to get $S-2rS+r^2S$.

The addition and subtraction of $S-2rS+r^2S$ for the S of equation (4.25) is illustrated in Table 3.

	$2r$	$+3r^2$	$+4r^3+5r^4$	$\ldots$	$+(n-1)r^{n-2}$	$+nr^{n-1}$		$:S$
		$-4r^2$	$-6r^3-8r^4$	$\ldots$	$-2(n-2)r^{n-2}$	$-2(n-1)r^{n-1}$	$-2nr^n$	$:-2rS$
	$2r$	$-r^2$	$-2r^3-3r^4$	$\ldots$	$-(n-3)r^{n-2}$	$-(n-2)r^{n-1}$	$-2nr^n$	$=S-2rS$
r^2S:			$+2r^3+3r^4$	$\ldots$	$+(n-3)r^{n-2}$	$+(n-2)r^{n-1}$	$+(n-1)r^n$	$+nr^{n+1}$
	$2r$	$-r^2$					$-(n+1)r^n$	$+nr^{n+1}$

TABLE 3

From the result of the operation shown in Table 3,

$$S-2rS+r^2S=S(r-1)^2=2r-r^2-(n+1)r^n+nr^{n+1}$$

multiplying by $\frac{-t_1}{(r-1)^2}$,

$$-t_1 S = -t_1 \left(\frac{2r - r^2 - (n+1)r^n + nr^{n+1}}{(r-1)^2} \right) = -t_1 \sum_{i=2}^{n} i r^{i-1} = c_1$$

The addition and subtraction of $S - 2rS + r^2 S$ for the S of equation (4.26) is illustrated in Table 4.

	r^2	$+2r^3$	$+3r^4 + 4r^5$	$\dots$	$+(n-2)r^{n-1}$	$+(n-1)r^n$		$:S$
		$-2r^3$	$-4r^4 - 6r^5$	$\dots$	$-2(n-3)r^{n-1}$	$-2(n-2)r^n$	$-2(n-1)r^{n+1}$	$:-2rS$
	r^2		$-r^4 - 2r^5$	$\dots$	$-(n-4)r^{n-1}$	$-(n-3)r^n$	$-2(n-1)r^{n+1}$	$= S - 2rS$
r^2S:			$+r^4 + 2r^5$	$\dots$	$+(n-4)r^{n-1}$	$+(n-3)r^n$	$+(n-2)r^{n+1}$	$+(n-1)r^{n+2}$
	r^2						$-nr^{n+1}$	$+(n-1)r^{n+2}$

TABLE 4

From the result of the operation shown in Table 4,

$$S - 2rS + r^2 S = S(r-1)^2 = r^2 - nr^{n+1} + (n-1)r^{n+2}$$

multiplying by $\frac{t_1}{(r-1)^2}$,

$$t_1 S = t_1 \left(\frac{r^2 - nr^{n+1} + (n-1)r^{n+2}}{(r-1)^2} \right) = t_1 \sum_{i=2}^{n} (i-1) r^i = c_0$$

□

The above proof is for any value of r. For the particular case of $r = 2$, equations (4.25) and (4.26) become

$$c_1 = -t_1 (n-1) 2^n \quad \text{and} \quad c_0 = t_1 (n-2) 2^{n+1} + 4 \tag{4.27}$$

Although the proof above is sufficient, we may prove these particular equations by mathematical induction. By the $k = 0$ case of equations (2.2) and (2.3), with $r = 2$ and $c_2, \dots, c_n = t_1$, we have

$$c_1 = -t_1 \sum_{i=2}^{n} i 2^{i-1} \quad \text{and} \quad c_0 = t_1 \sum_{i=2}^{n} (i-1) 2^i$$

Therefore, we need to show that

$$\text{(A)} \sum_{i=2}^{n} i 2^{i-1} = (n-1) 2^n \quad \text{and} \quad \text{(B)} \sum_{i=2}^{n} (i-1) 2^i = (n-2) 2^{n+1} + 4 \tag{4.28}$$

We verify that identity (4.28)(A) is true for $n = 2$. Assuming it is true for n, then,

$$\sum_{i=2}^{n+1} i 2^{i-1} = \sum_{i=2}^{n} i 2^{i-1} + (n+1) 2^{n+1-1} = (n-1) 2^n + (n+1) 2^n = 2n 2^n = n 2^{n+1} .$$

This is the same as the RHS of identity (4.28)(A) with $n+1$ replacing n. Hence, if the identity is true for n, then it is true for $n+1$. It is true for $n=2$, therefore it is also true for $n=2+1=3$, $n=3+1=4$, and so on.

To prove identity (4.28)(A),

$$\sum_{i=2}^{n+1}(i-1)2^i = \sum_{i=2}^{n}(i-1)2^i + (n+1-1)2^{n+1} = \left[(n-2)2^{n+1}+4\right] + n2^{n+1}\,.$$
$$= 2n2^{n+1} - 2^{n+2} + 4 = (n-1)2^{n+2} + 4\,.$$

This is the same as the RHS of identity (4.28)(B) with $n+1$ replacing n. We verify that it is true for $n=2$, then by induction it is true for integers $n \geq 2$.

Remark The value of c_0 in the $P_n(x)$ presented above, having 2 as a duplicate zero, is $4a_{n-1}$, where $a_n \in$ Sloane's number sequence A0003337, [13], which means $a_n = (n-1)2^n + 1 = \sum_{i=1}^{n} i2^{i-1}$. The value of c_1 in this same $P_n(x)$ is $-4b_{n-1}$, , where $b_n \in$ Sloane's number sequence A001787, [14], which means $b_n = n2^{n-1} = \sum_{i=1}^{n}(i+1)2^i$.

4.1. **A New Type of Geometric Progression.** Identity (4.25) presented in corollary 4.10 may be rewritten as

$$S_k = t_1\sum_{j=1}^{k} jr^{j-1} = \begin{cases} t_1\left(\frac{1-(k+1)r^k+kr^{k+1}}{(r-1)^2}\right) & if\ r \neq 1 \\ t_1\left(\frac{k(k+1)}{2}\right) & if\ r=1. \end{cases} \tag{4.29}$$

Equation (4.29) represents the sum of the first k terms of a new type of geometric progression, with the first term $= t_1$, the j^{th} term $= t_1jr^{j-1}$, and a ratio of succeeding terms, j and $j+1$, of $r\frac{j+1}{j}$. When $r<1$ in equation (4.29), $S_\infty = \frac{1}{(1-r)^2}$. Compare this to the common geometric progression with the first term $= t_1$, the j^{th} term $= t_1r^{j-1}$, a ratio of succeeding terms of r, and the sum of the first k terms $S_k = t_1\sum_{j=1}^{k} r^{j-1} = t_1\frac{r^n-1}{r-1}$ if $r \neq 1$, $S_k = kt_1$ if $r=1$. When $r<1$, $S_\infty = \frac{1}{(1-r)}$ for the common geometric progression.

For equation (4.26) When $r<1$, $S_\infty = \frac{r^2}{(1-r)^2}$.

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